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Introduction

In order that the flow in any laboratory model should be of value in interpreting or predicting the observed flow in the atmosphere, it is essential that the two systems should be dynamically similar. This means that it must be possible to describe the flow in the two systems by the same equations after appropriate adjustment of the units of length, time etc. has been made.

One of the difficulties which arise in attempting to devise a model which will be dynamically similar to the atmosphere is that air is a compressible fluid, and that in the adiabatic motion with which we are concerned the density changes with pressure so that

$$D/Dt\,(p/\rho^{\gamma}) = 0 .$$

Any model which is to be strictly similar must therefore use a compressible fluid with the same value of γ, and this restricts the models to those using air as a working fluid. Unfortunately, if the model is to be scaled down to laboratory size the similarity requirements lead to the need to use strong temperature gradients and very low velocities of a few centimetres per sec.; conditions which are impractical for useful experiment. The present note is intended to examine whether adequate similarity can be achieved using an incompressible fluid, and simulating the effects of atmospheric compressibility by modification of the shape of the model.

Extensive experiments have been carried out by Long (1954) on the flow of stably stratified liquids over obstacles in channels, although he has not attempted to simulate the important effects of wind shear. His results show vertical oscillations of the fluid closely analogous to those observed in the atmosphere and he has discussed the requirements for similarity between the flow in the atmosphere and the model. However, recent Meteorological Research Papers (M.R.P. 842 and 892) have tended to emphasise the limitations of model experiments and it seems desirable to restate the degree of similarity which may be achieved, in order that the possibilities of model experiments should not be overlooked.

The governing equations

The requirements for dynamical similarity in atmospheric motions have been discussed by Batchelor (1953) and his treatment can be followed here. The equations which govern atmospheric motion are assumed to be those which are applicable to the frictionless motion of a perfect gas under adiabatic conditions.

/They

They are the vector equation of motion

$$\frac{D\tilde{u}}{Dt} = \tilde{g} - \frac{1}{\rho}\tilde{\nabla}p, \qquad (1)$$

the equation of continuity

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla.\tilde{u}, \qquad (2)$$

and the relation between p and ρ for a perfect gas under adiabatic conditions, namely:-

$$\frac{D}{Dt}\left(p/\rho^{\gamma}\right) = 0. \qquad (3)$$

An essential feature of the type of motion with which we are concerned in the flow of air over mountains is that the pressure field is very nearly in hydrostatic equilibrium, and differs little from the pressure field in undisturbed flow. The density field also differs little from the density field of a fluid in adiabatic equilibrium. Batchelor has shown that these conditions lead to a simplification of the requirements for dynamical similarity and his treatment is followed here.

We write ρ_a and p_a for the density and pressure in an atmosphere with an adiabatic lapse rate at the same level z at which the true density and pressure are ρ and p. Under the circumstances outlined in the preceding paragraph any change in density of the moving air parcel arises almost entirely from the changes of pressure due to its change of level and these can be regarded as the same as those in the equilibrium atmosphere. Thus from (3)

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\gamma}\frac{1}{p}\frac{Dp}{Dt}$$

$$\simeq \frac{1}{\gamma}\frac{1}{p}w\frac{\partial p}{\partial z}$$

$$\simeq -\frac{g\rho_a}{\gamma p_a}w$$

Equation (2) thus becomes

$$\nabla.\tilde{u} = \frac{g\rho_a}{\gamma p_a}w \qquad (4)$$

Also

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\gamma p}\frac{Dp}{Dt} \simeq \frac{1}{\gamma p_a}\frac{Dp_a}{Dt} = \frac{1}{\rho_a}\frac{D\rho_a}{Dt}$$

Changing the variables in (1), (3) and (4) to $p - p_a$ and $\rho - \rho_a$ and making use of the fact that $\tilde{\nabla} p_a = \rho_a \tilde{g}$, these equations may be rewritten:-

$$\left(\frac{\rho-\rho_a}{\rho_a} + 1\right) \frac{D\tilde{u}}{Dt} = \tilde{g}\left(\frac{\rho-\rho_a}{\rho_a}\right) - \frac{\tilde{\nabla}(p-p_a)}{\rho_a}, \tag{5}$$

$$\nabla . \tilde{u} = \frac{g \rho_a}{\gamma p_a} w, \tag{6}$$

$$\frac{D}{Dt}\left(\frac{\rho-\rho_a}{\rho_a}\right) = 0 . \tag{7}$$

In order to write the equations in non-dimensional form, we introduce a unit length x_0, a unit velocity u_0, unit density ρ_0 and unit pressure p_0. The unit of time t_0 must then be given by $t_0 = x_0/u_0$.

Equations (5), (6) and (7) can then be written in non-dimensional form in terms of new variables, $u' = u/u_0$, etc., giving

$$\frac{D\tilde{u}'}{Dt'} = \frac{x_0 \rho_0}{u_0^2}\left(\frac{\rho'-\rho_a'}{\rho_a}\right)\tilde{g} - \frac{x_0 p_0}{u_0^2 \rho_a}\tilde{\nabla}(p'-p_a') \tag{8}$$

$$\nabla . \tilde{u}' = \frac{g \rho_a x_0}{\gamma p_a} w' \tag{9}$$

$$\frac{D}{Dt}\left(\frac{\rho'-\rho_a'}{\rho_a'}\right) = 0 \tag{10}$$

Here $\frac{\rho-\rho_a}{\rho_a}$ has been ignored compared with unity in equation (8).

An incompressible model of variable width

The essential difference between equations (8) - (10) which describe the atmosphere, and the equations which describe an incompressible fluid model arises from the equation of continuity (9). If, however, our study is restricted to two-dimensional motion in the atmosphere, it may be possible by using a narrow tank of variable width to bring the equation for the two-dimensional divergence to the form (9) but at the same time to retain the essentially two-dimensional character of the motion.

Thus if the width of the tank is $a = a(z)$ and the motion can still be regarded as essentially two-dimensional, the equation of continuity of an incompressible fluid becomes

$$\frac{\partial}{\partial x}(au) + \frac{\partial}{\partial z}(aw) = 0 ,$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = -\frac{w}{a}\frac{da}{dz} . \tag{11}$$

This is of a similar form to equation (9) provided the latter is treated as referring to two-dimensional motion, i.e.,

$$\nabla . u' = \frac{\partial u'}{\partial x'} + \frac{\partial w'}{\partial z'} .$$

The equations of motion for the two-dimensional motion of an incompressible fluid are:-

$$\frac{D\tilde{u}}{Dt} = \tilde{g} - \frac{1}{\rho} \nabla p , \tag{12}$$

and the equation corresponding to the equation of state (3) is

$$\frac{D\rho}{Dt} = 0 \tag{13}$$

If we write ρ_{am} and p_{am} as the density and pressure in a model with uniform density in hydrostatic equilibrium, the three equations (11), (12) and (13) become

$$\left(\frac{\rho - \rho_{am}}{\rho_{am}} + 1 \right) \frac{D\tilde{u}}{Dt} = \frac{\rho - \rho_{am}}{\rho_{am}} \tilde{g} - \frac{\nabla (p - p_{am})}{\rho_{am}} , \tag{14}$$

$$\nabla_2 . \tilde{u} = - \frac{w}{a} \frac{da}{dz} , \tag{15}$$

$$\frac{D}{Dt} \left(\frac{\rho - \rho_{am}}{\rho_{am}} \right) = 0 , \tag{16}$$

where $\nabla_2 .$ is the two-dimensional divergence.

/If

If again we neglect $(\rho - \rho_{am})/\rho_{am}$ compared with unity and express the equations in terms of non-dimensional variables by means of unit lengths, velocities etc. x_m, u_m, we have

$$\frac{D\tilde{u}'}{Dt'} = \frac{x_m \rho_m}{u_m^2 \rho_{am}} (\rho' - \rho'_{am}) \tilde{g} - \frac{x_m p_m}{u_m^2 \rho_{am}} \nabla (p' - p_{am}) \qquad (17)$$

$$\nabla_2 . \tilde{u}' = - \frac{x_m}{a} \frac{da}{dz} w' \qquad (18)$$

$$\frac{D}{Dt} \left(\frac{\rho' - \rho'_{am}}{\rho'_{am}} \right) = 0 \qquad (19)$$

Conditions for dynamical similarity

The equations (17), (18) and (19) can be made identical with (8), (9) and (10) provided the following conditions are satisfied.

$$\frac{g \rho_a x_o}{\gamma p_a} = - \frac{x_m}{a} \frac{da}{dz} , \qquad (20)$$

$$\frac{x_o \rho_o}{u_o^2 \rho_a} = \frac{x_m}{u_m^2} \frac{\rho_m}{\rho_{am}} , \qquad (21)$$

$$\frac{x_o p_o}{u_o^2 \rho_a} = \frac{x_m}{u_m^2} \frac{p_m}{\rho_{am}} . \qquad (22)$$

It should be noted that both ρ_o and ρ_a are functions of z but equations (21) and (22) ensure that $\rho' = \rho_o/\rho_a$ is a constant and that the coefficients in equation (8) are also constants.

The third relation determines the ratio between the pressure units but as pressure will not be specified in the initial or boundary conditions (being given adequately by the hydrostatic condition) this need not concern us here.

Equation (20) shows that

$$\frac{da}{a} = - \frac{g \rho_a x_o}{\gamma p_a x_m} dz ,$$

where dz is a height increment in the model.

If dz^* is a corresponding height increment in the atmosphere

$$dz^* = \frac{x_o}{x_m} dz \quad .$$

Thus $$\int \frac{da}{a} = -\int \frac{g\rho_a}{\gamma p_a} dz = \frac{1}{\gamma}\int \frac{dp}{p_a} \simeq \frac{1}{\gamma}\int \frac{dp_a}{p_a} ,$$

or $$a \simeq A p_a^{1/\gamma} = B\rho_a \quad .$$

Thus equation (20) implies that the width of the model shall be proportional to the density at the corresponding level in the atmosphere. Equation (21) is the only remaining condition to be satisfied namely

$$\frac{x_o \rho_o}{u_o^2 \rho_a} = \frac{x_m \rho_m}{u_m^2 \rho_{am}} \quad . \qquad (21)$$

ρ_o/ρ_a and ρ_m/ρ_{am} measure the ratio of the density unit (or density gradient) to the density itself in the atmosphere and the model respectively. If these proportional gradients are the same and $x_o/u_o^2 = x_m/u_m^2$ the neglect of $(\rho-\rho_a)/\rho_a$ in (5) and the similar approximation in deriving (17) from (14) are unnecessary. Equation (21) expresses the equality of the Froude number in the model and the atmosphere (see Long, 1954).

The physical requirements of a fluid model of two-dimensional flow over a ridge

In the atmosphere a typical range of potential temperature from the 1000 mb. to the 500 mb. level is around 12% so the change in $(\rho-\rho_a)/\rho_a$ is about 12% through the layer. If ρ_o/ρ_a is to equal ρ_m/ρ_{am} a similar proportional change in density is required through the model. Such a density gradient might be achieved with salt solution of varying concentration but only about a quarter of the change could conveniently be achieved in a liquid model using temperature gradients. In which case we should have

$$\rho_o/\rho_a = 4\rho_m/\rho_{am} \quad .$$

Thus the restrictions on the dimensions and velocities are

$$\frac{x_o}{x_m}\cdot\frac{u_m^2}{u_o^2} = 1 \text{ for a salt solution model}$$

$$= \tfrac{1}{4} \text{ for a temperature gradient model.}$$

If a model about 2 ft. high is contemplated, $x_o/x_m = 10^4$

and $u_m/u_o = 10^{-2}$ for a salt solution model

$= 0{\cdot}5 \times 10^{-2}$ for a temperature gradient model.

/Taking

Taking u_0 in the atmosphere in the range 5 to 50 m/s this requires flow velocities in the models of 2·5 to 50 cm./sec. Other requirements already considered are that the flow should take place in a deep narrow channel with a width proportional to the density at a corresponding level in a standard atmosphere.

Boundary conditions

In constructing a model of the type described the lower boundary presents no difficulty. Similarity in shape between the model and the hill on the appropriate scale is all that is required. It can also be assumed that if the flow is constrained to be horizontal at a distance upstream from the model ridge the lateral boundary conditions will be similar to those in the atmosphere.

The upper boundary of the model cannot however satisfactorily reproduce conditions in the atmosphere; to be truly similar the model would need to be of infinite depth, tapering continuously with increasing height. However, it seems unlikely that the conditions in the upper atmosphere strongly influence the flow at low levels, and it may be possible to demonstrate this experimentally in a model. The atmospheric flow might be expected to be intermediate between that appropriate to a model with a rigid lid and one with a free surface, and if these showed little difference in the lower sections of the model it might reasonably be inferred that the upper boundary condition was not important.

Viscosity

Neither in the model nor in the atmosphere is it expected that viscous effects are of primary importance except in a shallow boundary layer. In the model, the shears to be expected in the body of the fluid are of the order of 50 cm./sec. in 60 cm. and with viscosity of ·013 c.g.s. units appropriate to water, the tangential stresses would be of the order of ·01 dynes/cm.^{2}. If this changes to zero over a distance of 10 cm. the net acceleration of the fluid would be 10^{-3} cm./sec.^{2}. However the accelerations, Du/Dt , are likely to be of the order of 1-100 cm./sec.^{2} so that viscous effects should be unimportant away from the boundary layer.

Since the kinematic viscosity has dimensions l^2t^{-1} or lu the value of ·013 appropriate to water would correspond to a value of $1·3 \times 10^4$ in the atmosphere if the salt solution model considered earlier were used, or $2·6 \times 10^4$ if the temperature gradient model were used. These values are of much the same order as the atmospheric eddy viscosity in the boundary layer and the effect of molecular viscosity in the model may be somewhat like that of eddy viscosity in the atmosphere.

Summary and recommendation

From the preceding analysis it appears that a useful degree of similarity can be achieved between the flow of liquid in a channel and the flow of air over a ridge of hills and that experiments with fluid models would be valuable. It is essential that the fluid should contain a density and velocity gradient because these have been shown by R.S. Scorer (1949) and others to be the primary factors which determine the nature of the flow. The effect of compressibility in the atmosphere might be simulated by making the channel taper upwards, the width being proportional to the density at a corresponding level in the atmosphere.

References

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